

Rate of convergence to self-similarity for the fragmentation equation in L^1 spaces

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Abstract

In a recent result by the authors [1] it was proved that solutions of the self-similar fragmentation equation converge to equilibrium exponentially fast. This was done by showing a spectral gap in weighted L^2 spaces of the operator defining the time evolution. In the present work we prove that there is also a spectral gap in weighted L^1 spaces, thus extending exponential convergence to a larger set of initial conditions. The main tool is an extension result in [4].

1 Introduction

In a recent paper [1] we have studied the speed of convergence to equilibrium for solutions of equations involving the fragmentation operator and first-order differential terms. In this paper we will focus on the case of self-similar fragmentation given by

$$\partial_t g_t(x) = -x\partial_x g_t(x) - 2g_t(x) + \mathcal{L}g_t(x) \quad (1.1a)$$

$$g_0(x) = g_{in}(x) \quad (x > 0). \quad (1.1b)$$

Here the unknown is a function $g_t(x)$ depending on time $t \geq 0$ and on size $x > 0$, which represents a density of units (usually particles, cells or polymers) of size x at time t , and g_{in} is an initial condition. The *fragmentation operator* \mathcal{L} acts on a function $g = g(x)$ as

$$\mathcal{L}g(x) := \mathcal{L}_+g(x) - B(x)g(x), \quad (1.2)$$

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where the positive part \mathcal{L}_+ is given by

$$\mathcal{L}_+g(x) := \int_x^\infty b(y, x)g(y) dy. \quad (1.3)$$

The coefficient $b(y, x)$, defined for $y > x > 0$, is the *fragmentation coefficient*, and $B(x)$ is the *total fragmentation rate* of particles of size $x > 0$. It is obtained from b through

$$B(x) := \int_0^x \frac{y}{x} b(x, y) dy \quad (x > 0). \quad (1.4)$$

We refer to [1, 5, 7, 2, 6, 8] for a motivation of (1.1) in several applications and a general survey of the mathematical literature related to it.

We call T the operator on the right hand side of (1.1a), this is,

$$Tg(x) := -x\partial_x g(x) - 2g(x) + \mathcal{L}g(x) \quad (x > 0), \quad (1.5)$$

acting on a (sufficiently regular) function g defined on $(0, +\infty)$. Notice that, even though g is a one-variable function, we still denote its derivative as $\partial_x g$ in order to be consistent with the notation in (1.1). The results in [1] show that T has a spectral gap in the space $L^2(xG^{-1})$, where G is the unique stationary solution of (1.1) with $\int xG = 1$. In the rest of this paper G will represent this solution, called the *self-similar profile*. Proofs of existence of the profile G and some estimates are given in [3, 6, 2], and additional bounds are given in [1].

The main result in [1] is a study of the long time behavior of (1.1): by means of an inequality relating the quadratic entropy and its dissipation rate, exponential convergence is obtained in $L^2(xG^{-1})$. Using the results in [4] this is further extended to the space $L^2(x + x^k)$ for a sufficiently large exponent k . In this way one obtains convergence in a strong norm, but correspondingly has to impose more on the initial condition than just having finite mass.

The purpose of this work is to prove that T has a spectral gap in the larger spaces $L^1(x^m + x^M)$, where $1/2 < m < 1 < M$ are suitable exponents. This extension is an example of application of the results in [4]. The interest of this concerning the asymptotic behavior of (1.1) is that it shows exponential convergence is valid for more general initial conditions (any function in $L^1(x^m + x^M)$).

Assumptions on the fragmentation coefficient In order to use the results in [1] we will make the following hypotheses on the fragmentation coefficient b :

Hypothesis 1.1. *For all $x > 0$, $b(x, \cdot)$ is a nonnegative measure on the interval $[0, x]$. Also, for all $\psi \in \mathcal{C}_0([0, +\infty))$, the function $x \mapsto \int_{[0, x]} b(x, y)\psi(y) dy$ is measurable.*

Hypothesis 1.2. *There exists $\kappa > 1$ such that*

$$\int_0^x b(x, y) dy = \kappa B(x) \quad (x > 0). \quad (1.6)$$

Hypothesis 1.3. *There exists $0 < B_m < B_M$ satisfying*

$$2B_m x^{\gamma-1} \leq b(x, y) \leq 2B_M x^{\gamma-1} \quad (0 < y < x) \quad (1.7)$$

for some $0 < \gamma < 2$.

This implies the following useful bound, as remarked in [1, Corollary 6.4]:

Lemma 1.4. *Consider a fragmentation coefficient b satisfying Hypotheses 1.1–1.3. There exists a strictly decreasing function $k \mapsto p_k$ for $k \geq 0$ with $\lim_{k \rightarrow +\infty} p_k = 0$,*

$$p_k > 1 \text{ for } k \in [0, 1), \quad p_1 = 1, \quad 0 < p_k < 1 \text{ for } k > 1, \quad (1.8)$$

and such that

$$\int_0^x y^k b(x, y) dy \leq p_k x^k B(x) \quad (x > 0, k > 0). \quad (1.9)$$

Main results The main result of the present work is a spectral gap of T on weighted L^1 spaces.

Theorem 1.5. *Assume hypotheses 1.1–1.3. For any $1/2 < m < 1$ there exists $1 < M < 2$ such that the operator (1.5) has a spectral gap in the space $X := L^1(x^m + x^M)$. More precisely, there exists $\alpha > 0$ and a constant $C \geq 1$ such that, for all $g_{in} \in X$ with $\int x g_{in} = 1$*

$$\|g_t - G\|_X \leq C e^{-\alpha t} \|g_{in} - G\|_X \quad (t \geq 0).$$

2 Preliminaries

In this section we gather some known results from previous works.

2.1 Previous results on the spectral gap of T

A result like Theorem 1.5 was proved in [1], but in the L^2 space with weight $x G^{-1}$. This is summarized in the following theorem:

Theorem 2.6 ([1]). *Assume Hypotheses 1.1–1.3, and consider G the self-similar profile with $\int x G = 1$. The operator T given by (1.5) has a spectral gap in the space $H = L^2(x G^{-1})$.*

More precisely, there exists $\beta > 0$ such that for any $g_{in} \in H$ with $\int x g = 1$ the solution $g \in C([0, \infty); L^1(x dx))$ to equation (1.1) satisfies

$$\|g_t - G\|_H \leq e^{-\beta t} \|g_{in} - G\|_H \quad (t \geq 0).$$

2.2 Bounds for the self-similar profile

We recall the following result from [1, Theorem 3.1]:

Theorem 2.7. *Assume Hypotheses 1.1–1.3 on the fragmentation coefficient b , and call $\Lambda(x) := \int_0^x \frac{B(s)}{s} ds$. Let G be the self-similar profile with $\int x G = 1$.*

For any $\delta > 0$ and any $a \in (0, B_m/B_M)$, $a' \in (1, +\infty)$ there exist constants $C' = C'(a', \delta)$, $C = C(a) > 0$ such that

$$C' e^{-a' \Lambda(x)} \leq G(x) \leq C e^{-a \Lambda(x)} \quad \text{for } x > 0. \quad (2.10)$$

Remark 2.8. *In the case $b(x, y) = 2x^{\gamma-1}$ (so $B(x) = x^\gamma$), the profile G has the explicit expression $G(x) = e^{-\frac{x^\gamma}{\gamma}}$ for $\gamma > 0$. This motivates the choice of $e^{-a \Lambda(x)}$ as functions for comparison. For a general $b(x, y)$ no explicit form is available.*

Proof. Everything but the lower bound of G for small x is proved in [1, Section 3]. For the lower bound, we calculate as follows:

$$\partial_x \left(x^2 e^{\Lambda(x)} G(x) \right) = x e^{\Lambda(x)} \int_x^\infty b(y, x) G(y) dy \quad (x > 0), \quad (2.11)$$

which implies that $x^2 e^{\Lambda(x)} G(x)$ is a nondecreasing function. Hence, it must have a limit as $x \rightarrow 0$, and this limit must be 0 since we know $x G(x)$ is integrable. Then, integrating (2.11), and for $0 < z < 1$,

$$\begin{aligned} z^2 e^{\Lambda(z)} G(z) &= \int_0^z x e^{\Lambda(x)} \int_x^\infty b(y, x) G(y) dy dx \\ &= \int_0^\infty G(y) \int_0^{\min\{z, y\}} b(y, x) x e^{\Lambda(x)} dx dy \\ &\geq 2B_m \int_0^\infty y^{\gamma-1} G(y) \int_0^{\min\{z, y\}} x dx dy \\ &= B_m \int_0^\infty y^{\gamma-1} G(y) (\min\{z, y\})^2 dy \\ &\geq B_m z^2 \int_z^\infty y^{\gamma-1} G(y) dy \\ &\geq B_m z^2 \int_1^\infty y^{\gamma-1} G(y) dy = C z^2 \quad (0 < z < 1). \end{aligned} \quad (2.12)$$

Notice that the number $\int_1^\infty y^{\gamma-1} G(y) dy$ is strictly positive, as the profile G is strictly positive everywhere (see [2, 3, 1]). This proves the lower bound on $G(x)$ for $0 < x < 1$, and completes the proof. \square

2.3 A general spectral gap extension result

Our proof is based on the following result from [4], which was already used in [1] for an extension to an L^2 space with a polynomial weight:

Theorem 2.9. *Consider a Hilbert space H and a Banach space X (both over the field \mathbb{C} of complex numbers) such that $H \subset X$ and H is dense in X . Consider two unbounded closed operators with dense domain T on H , Λ on X such that $\Lambda|_H = T$. On H assume that*

1. *There is $G \in H$ such that $TG = 0$ with $\|G\|_H = 1$;*
2. *Defining $\psi(f) := \langle f, G \rangle_H$, the space $H_0 := \{f \in H; \psi(f) = 0\}$ is invariant under the action of T .*
3. *$T - a$ is dissipative on H_0 for some $a < 0$, in the sense that*

$$\forall g \in D(T) \cap H_0 \quad ((T - a)g, g)_H \leq 0,$$

where $D(T)$ denotes the domain of T in H .

4. *T generates a semigroup e^{tT} on H ;*
Assume furthermore on X that
5. *there exists a continuous linear form $\Psi : X \rightarrow \mathbb{R}$ such that $\Psi|_H = \psi$;*
and Λ decomposes as $\Lambda = \mathcal{A} + \mathcal{B}$ with
6. *\mathcal{A} is a bounded operator from X to H ;*
7. *\mathcal{B} is a closed unbounded operator on X (with same domain as $D(\Lambda)$ the domain of Λ) and the semigroup $e^{t\mathcal{B}}$ it generates satisfies, for some constant $C \geq 1$, that*

$$\forall t \geq 0, \forall g \in X \text{ with } \Psi(g) = 0, \quad \|e^{t\mathcal{B}}g\|_X \leq C\|g\|_X e^{at}, \quad (2.13)$$

where $a < 0$ is the one from point 3.

Then, for any $a' \in (a, 0)$ there exists $C_{a'} \geq 1$ such that

$$\forall t \geq 0, \forall g \in X, \quad \|e^{t\Lambda}g - \Psi(g)G\|_X \leq C_{a'}\|g - \Psi(g)G\|_X e^{a't}.$$

3 Proof of the main theorem

The proof consists is an application of Theorem 2.9. For this, we consider the Hilbert space $H := L^2(xG^{-1}(x))$, where G is the unique self-similar profile with $\int G = 1$, and define $\psi(g) := \int xg$. Due to our previous results [1] we know that T and ψ satisfy points 1–4 of Theorem 2.9.

As the larger space we take $X = L^1(x^m + x^M)$, with $1/2 < m < 1 < M$, to be precised later. Observe that, due to the bounds on G from Theorem 2.7,

$$\begin{aligned} \|g\|_X &= \int_0^\infty (x^m + x^M)|g(x)| dx \\ &\leq \left(\int_0^\infty g(x)^2 \frac{x}{G(x)} dx \right)^{1/2} \left(\int_0^\infty (x^{m-\frac{1}{2}} + x^{M-\frac{1}{2}})^2 G(x) dx \right)^{1/2} = C\|g\|_H, \end{aligned}$$

and hence $H \subseteq X$. Similarly,

$$\int_0^\infty x|g(x)| dx \leq \int_0^\infty (x^m + x^M)|g(x)| dx,$$

which allows us to define $\Psi : X \rightarrow \mathbb{R}$, $\Psi(g) := \int xg$, and proves that Ψ is continuous on X . Obviously $\Psi|_H = \psi$, so point 5 of Theorem 2.9 is also satisfied.

Consider Λ the unbounded operator on X given by the same expression (1.5) (with domain a suitable dense subspace of X which makes Λ a closed operator). To prove the remaining points 6 and 7 we use the following splitting of Λ , taking real numbers $0 < \delta < R$ to be chosen later:

$$\begin{aligned} \mathcal{A}g(x) &:= \mathcal{L}^{+,s}g(x) := \int_x^\infty b_{R,\delta}(y,x) g(y) dy \\ &= \mathbf{1}_{x \leq R} \int_x^\infty \mathbf{1}_{y \geq \delta} b(y,x) g(y) dy, \end{aligned} \quad (3.14)$$

$$\Lambda = \mathcal{A} + \mathcal{B}, \quad \mathcal{B}g := \Lambda g - \mathcal{A}g, \quad (3.15)$$

where we denote $b_{R,\delta}(x,y) := b(x,y) \mathbf{1}_{x \geq \delta} \mathbf{1}_{y \leq R}$. We define

$$\begin{aligned} \mathcal{L}^{+,r}g &:= \mathcal{L}^+g - \mathcal{L}^{+,s}g \\ &= \int_x^\infty b(y,x) (1 - \mathbf{1}_{y \geq \delta} \mathbf{1}_{x \leq R}) g(y) dy \\ &= \int_x^\infty b(y,x) \mathbf{1}_{y \leq \delta} g(y) dy + \int_x^\infty b(y,x) \mathbf{1}_{y \geq \delta} \mathbf{1}_{x \geq R} g(y) dy \\ &=: \mathcal{L}_1^{+,r}g + \mathcal{L}_2^{+,r}g \end{aligned}$$

so we may write \mathcal{B} as

$$\mathcal{B}g = -2g - x\partial_x g - \mathcal{B}g + \mathcal{L}_1^{+,r}g + \mathcal{L}_2^{+,r}g. \quad (3.16)$$

First, let us prove that \mathcal{A} is bounded from X to H . We compute

$$\begin{aligned} \|\mathcal{A}g\|_H^2 &= \int_0^\infty x (\mathcal{L}^{+,s}g)^2 G(x)^{-1} dx \\ &\leq (2B_M)^2 \left(\sup_{[0,R]} x G(x)^{-1} \right) \int_0^R \left(\int_{\max(x,\delta)}^\infty y^{\gamma-1} g(y) dy \right)^2 dx \\ &\leq C_R \left(\int_\delta^\infty y^{\gamma-1} g(y) dy \right)^2 \leq C_{R,\delta} \left(\int_0^\infty y g(y) dy \right)^2 \leq C_{R,\delta} \|g\|_X^2, \end{aligned}$$

which shows $\mathcal{A} : X \rightarrow H$ is a bounded operator. Notice that we have used here the lower bound $G(x) \geq Cx$ for x small, proved in Theorem 2.7.

Then, let us prove that one can choose $0 < \delta < R$ appropriately so that \mathcal{B} satisfies point 7 of Theorem 2.9 for some $a < 0$. It is enough to prove that, for g a real function in the domain of Λ (the same as the domain of \mathcal{B}),

$$\int_0^\infty \text{sign}(g(x)) \mathcal{B}g(x) (x^m + x^M) dx \leq a \|g\|_X, \quad (3.17)$$

since then one can obtain (2.13) with $C = 1$ by considering the time derivative of the L^1 norm of $e^{t\mathcal{B}}g$. If we have this for any real g , it is easy to show it also holds for a complex g and some constant $C \geq 1$. So, we take g real and in the domain of Λ , and calculate as follows for any $k > 0$, using (3.16):

$$\begin{aligned} \int_0^\infty \text{sign}(g(x)) \mathcal{B}g(x) x^k dx &\leq (k-1) \int_0^\infty x^k |g| dx \\ &\quad - \int_0^\infty B(x) x^k |g| dx + \int_0^\infty |\mathcal{L}_1^{+,r} g| x^k dx + \int_0^\infty |\mathcal{L}_2^{+,r} g| x^k dx, \end{aligned} \quad (3.18)$$

where the first term is obtained from the terms $-2g - \partial_x g$ through an integration by parts. We give separately some bounds on the last two terms in (3.18). On one hand, we have

$$\begin{aligned} \int_0^\infty |\mathcal{L}_1^{+,r} g| x^k dx &\leq \int_0^\infty x^k \int_x^\infty b(y, x) \mathbf{1}_{y \leq \delta} |g(y)| dy dx \\ &\leq \int_0^\delta |g(y)| \left(\int_0^y x^k b(y, x) dx \right) dy \\ &\leq 2B_M \int_0^\delta |g(y)| B(y) y^k dy \\ &\leq p_k B_m \delta^\gamma \int_0^\delta |g(y)| y^k dy, \end{aligned} \quad (3.19)$$

where we have used (1.9). On the other hand, and again due to (1.9),

$$\begin{aligned} \int_0^\infty |\mathcal{L}_2^{+,r} g| x^k dx &\leq \int_0^\infty x^k \int_x^\infty b(y, x) \mathbf{1}_{x \geq R} \mathbf{1}_{y \geq \delta} |g(y)| dy dx \\ &\leq \int_0^\infty x^k \int_x^\infty b(y, x) \mathbf{1}_{x \geq R} \mathbf{1}_{y \geq R} |g(y)| dy dx \\ &\leq \int_R^\infty |g(y)| \left(\int_R^y x^k b(y, x) dx \right) dy \\ &\leq p_k \int_R^\infty |g(y)| y^k B(y) dy. \end{aligned} \quad (3.20)$$

Hence, from (3.18) and the bounds (3.19)–(3.20) we obtain

$$\begin{aligned}
& \int_0^\infty \mathcal{B}g(x) \operatorname{sign}(g(x))(x^m + x^M) dx \\
& \leq (m-1) \int_0^\infty x^m |g| dx + (M-1) \int_0^\infty x^M |g| dx \\
& \quad - \int_0^\infty B(x)(x^m + x^M) |g| dx \\
& \quad + p_m B_m \delta^\gamma \int_0^\delta x^m |g(x)| dx + p_m \int_R^\infty x^m B(x) |g(x)| dx \\
& \quad + p_M B_m \delta^\gamma \int_0^\delta x^M |g(x)| dx + p_M \int_R^\infty x^M B(x) |g(x)| dx. \quad (3.21)
\end{aligned}$$

We have to choose $1/2 < m < 1 < M < 2$ so that this is bounded by $-C\|g\|_X$ for some positive constant C . First, fix any m with $1/2 < m < 1$, and take $0 < \delta < 1$ small enough such that

$$p_m B_m \delta^\gamma < \frac{1-m}{4}, \quad B_m \delta^\gamma < \frac{1-m}{4}.$$

(Which can be done due to $\gamma > 0$.) Then, as $p_M < 1$ and $x^M < x^m$ for $x < \delta < 1$,

$$\begin{aligned}
& \int_0^\infty \mathcal{B}g(x) \operatorname{sign}(g(x))(x^m + x^M) dx \\
& \leq -\frac{1-m}{2} \int_0^\infty x^m |g| dx + (M-1) \int_0^\infty x^M |g| dx \\
& \quad - \int_0^\infty B(x)(x^m + x^M) |g| dx \\
& \quad + p_m \int_R^\infty x^m B(x) |g(x)| dx + p_M \int_R^\infty x^M B(x) |g(x)| dx. \quad (3.22)
\end{aligned}$$

Now, take $R_0 > 0$ such that $B(x) > 2 > M$ for $x \geq R_0$. Then, choose $1 < M < 2$ such that $(M-1)x^M < \frac{1-m}{4}x^m$ for $0 < x < R_0$. Then whatever R is we have from (3.21):

$$\begin{aligned}
& \int_0^\infty \mathcal{B}g(x) \operatorname{sign}(g(x))(x^m + x^M) dx \\
& \leq -\frac{1-m}{4} \int_0^{R_0} x^m |g| dx - \int_{R_0}^R x^M |g| dx \\
& \quad - \int_R^\infty (B(x) - M + 1)x^M |g| dx \\
& \quad + p_m \int_R^\infty x^m B(x) |g(x)| dx + p_M \int_R^\infty x^M B(x) |g(x)| dx. \quad (3.23)
\end{aligned}$$

Finally, choose $R > 1$ such that

$$-(B(x)(1 - p_M) - M + 1)x^M + p_m x^m \leq -x^M \quad \text{for } x > R.$$

With this, and continuing from (3.23),

$$\begin{aligned} & \int_0^\infty \mathcal{B}g(x) \operatorname{sign}(g(x))(x^m + x^M) dx \\ & \leq -\frac{1-m}{4} \int_0^{R_0} x^m |g| dx - \int_{R_0}^\infty x^M |g| dx \\ & \leq -C \|g\|_X, \end{aligned} \quad (3.24)$$

for some number $C = C(m, M, R_0) > 0$. This shows point that \mathcal{B} is dissipative with constant $-C$, and hence point 7 of Theorem 2.9 holds with $a = -C$. A direct application of Theorem 2.9 then proves our result, Theorem 1.5, with $\alpha := \min\{\beta, C\}$ (where β is the one appearing in Theorem 2.6).

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